P-Median Problem with Fuzzy Demands and Vertices

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Abstract

In this paper, we study *p*-median problem when vertices and demands are fuzzy. First, we consider the problem as a fuzzy vertex problem and then apply the yield network for fuzzy demand problem. In the end, we define membership degree for fuzzy vertex and demand problem.

Keywords: P-Median, Fuzzy, Demand

1. Introduction

The *p*-median problem is one of the important problems in location theorem. The goal of *p*-median problem on network is finding set $X = \{ x_1, ..., x_p \}$ consisting of location of facility on $N = (V, E)$ network such that the sum of distances of vertices of network from X is minimum. Distance of $v \in V$ from X is distance between v and nearest vertex in X. In general, weight *wi* allocated to vertex v_i and thus goal function is

$$\min F(X) = \min \sum_{v_i \in V} w_i d(X, v_i)$$

It is well known that the crisp *p*-median problem can be modeled by a mixed integer 0-1 problem [1, 2]:

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} d_{ij}$$

$$s.t \ \sum_{i=1}^{n} x_{ij} = w_j, \quad 1 \le j \le n \qquad (1)$$

$$0 \le x_{ij} \le w_j y_i, \ 1 \le j, j \le n \qquad (2)$$

$$\sum_{i=1}^{n} y_i = p,$$

$$y_i \in \{0,1\}, \qquad 1 \le i \le n, \qquad (3)$$

where x_{ij} is the demand of vertex v_j covered by the facility at vertex v_i and 0, therwise, w_i is the demand at vertex v_i and d_{ij} is the distance from v_i to v_j.

Fuzzy *p*-median problem stated four cases: fuzzy vertices, fuzzy edge, fuzzy demand and fuzzy distance or composed this cases.

2. *p*-median Problem with Fuzzy Demands and Vertices

Definition 1. Assume that we given a fuzzy goal $\tilde{G}$ and a fuzzy constraint $\tilde{C}$ in a space of alternatives *X*. Then $\tilde{G}$ and $\tilde{C}$ combine to form a decision , $\tilde{D}$ which is a fuzzy set resulting from intersection of $\tilde{G}$ and $\tilde{C}$, that is, $\tilde{D} = \tilde{G} \cap \tilde{C}$ and correspondingly, $\mu_{\tilde{D}} = \min\{\mu_{\tilde{G}}, \mu_{\tilde{C}}\}$

Note that the intersection of fuzzy sets is defined in the possible sense by the Min operator [6]. If we want a final crisp decision, then we look for a solution where $\mu_{\tilde{D}}$ is maximum. We definite μ_f (degree of feasibility) and μ_g (degree of improvement of the goal) for each solution (x_{ij}, y_i) as

$$\mu_f = (x_{ij}, y_i) = h_f\left(\sum_{j=1}^{n} w_j - \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}\right) \qquad (4)$$

where h_f is an auxiliary function given by

$$h_f(x) = \begin{cases} 1 & x < 0 \\ 1 - \dfrac{x}{p_f} & 0 \le x \le p_f \\ 0 & x > p_f \end{cases} \qquad (5)$$

where p_f represents the maximum tolerance level for a solution may be considered feasible. Also,

$$\mu_g = (x_{ij}, y_i) = h_g\left(Z^* - \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}\right) \qquad (6)$$

where Z^* is crisp optimum cost, and h_g is another auxiliary function given by

$$h_g(x) = \begin{cases} 1 & x < 0 \\ \dfrac{x}{p_g} & 0 \le x \le p_g \\ 0 & x > p_g \end{cases} \qquad (7)$$

where p_g indicates how much the cost should be reduced in order for the improvement to be considered as completely satisfactory. Therefore, fuzzy *p*-median problem is as:

Find (x_{ij}, y_i)

$$\begin{aligned} & \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} d_{ij} \le Z^* \\ & \sum_{i=1}^{n} x_{ij} \le w_j \qquad \forall j \in \{1,2,...,n\} \\ & \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} > \sum_{j=1}^{n} w_j \\ & 0 \le x_{ij} \le w_j y_i \qquad 1 \le i, j \le n \\ & \sum_{i=1}^{n} y_i = p \qquad y_i \in \{0,1\} \quad 1 \le i \le n \end{aligned} \qquad (8)$$

Assume that λ is the membership degree to the decision set. $\tilde{D}$ On the other hand, λ is the global degree of satisfaction of a given solution. Therefore, we can find solutions of the greatest value with the following auxiliary crisp problem:

$$\begin{aligned} & \max \lambda \\ & s.t \quad \lambda \le \mu_g = (x_{ij}, y_i) \\ & \qquad \lambda \le \mu_f = (x_{ij}, y_i) \\ & \qquad 0 \le x_{ij} \le w_j y_i \qquad 1 \le j, j \le n \\ & \qquad \sum_{i=1}^{n} x_{ij} \le w_j \qquad \forall j \in \{1,2,...,n\} \\ & \qquad \sum_{i=1}^{n} y_i = p \qquad y_i \in \{0,1\} \\ & \qquad 1 \le i \le n \qquad 0 \le \lambda \le 1 \end{aligned} \qquad (9)$$

If we replace μ_g and μ_f by their definitions, then above auxiliary crisp problem will as:

$$
\begin{aligned}
&\max \lambda \\
&s.t \;\; \lambda + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{c_{ij} d_{ij}}{p_g} x_{ij} \le \frac{Z^*}{p_g} \\
&\lambda + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{p_g} x_{ij} \le 1 - \frac{\sum_{j=1}^{n} w_j}{p_f} \\
&0 \le x_{ij} \le w_j y_i \qquad 1 \le j, j \le n \\
&\sum_{i=1}^{n} x_{ij} \le w_j \qquad \forall j \in \{1,2,\dots,n\} \\
&\sum_{i=1}^{n} y_i = p \qquad y_i \in \{0,1\} \\
&1 \le i \le n \qquad 0 \le \lambda \le 1
\end{aligned}
$$

By attention described algorithm in [3], we dissolve this model near $x = 0$. At first, we mention some definition.

Now, for each possible reduction x of the demand, we represent the estimated benefit by a tolerance interval $[a(x), b(x)]$. We can assume that $a(x)$ and $b(x)$ are continuous functions of x, so they determine a subset that we will call *gross profit band*. Define $p_\lambda(x) = (1-\lambda)a(x) + \lambda b(x) \qquad 0 \le \lambda \le 1$

Hence, λ can be seen as an "optimum level" for the possible profits for each covered demand. In order to compare p_λ and the cost function R, it is useful to analyze their average rate of decrease in the following sense:

Definition 2. We call the average rate of decrease of the cost to the function

$$S^p(x) = \frac{R(x) - R(0)}{x} \qquad x > 0$$

where $R(x)$ is decrease function of demand.

Definition 3. The average rate of decrease of the profits for each level λ is the Function

$$S_\lambda{}^p(x) = \frac{R_\lambda(x) - R_\lambda(0)}{x} \qquad x > 0$$

If we consider $p(x)$ crisp, then we have $S^p(x) > S^c(x) \quad S^p(x) \le S^c(x)$ [4]

However, we are considering a fuzzy estimation of the profits, and this implies that the crisp interval $S^c(x) \in [S^p(x), \infty[$ has now an uncertain extreme. Since

$$S_\lambda^p(x) = (1-\lambda)S_0^p(x) + \lambda S_1^p(x) \qquad 0 \le \lambda \le 1$$

we have the possible extreme $S_\lambda^p(x)$ may oscillate between

$$m(x)=\min\{S_0^p(x),S_1^p(x)\} \qquad M(x)=\max\{S_0^p(x),S_1^p(x)\}$$

Hence, the fuzzy analogous to the interval $S^c(x)\in[S^p(x),\infty[$ is the fuzzy subset given by the following membership function:

$$\mu_x(y)=\begin{cases}1 & y>M(x)\\ \dfrac{y-m(x)}{M(x)-m(x)} & m(x)\le y\le M(x)\\ 0 & y<m(x)\end{cases}$$

Notice that for simplicity, we have chosen a piecewise linear membership. function. Now, reasoning as before, we see that $\mu(S^c(x))=0$ for a certain value of x means that reducing the covered demand by x units implies an increase in the net profit and hence the presence of a virtual loss. On the other hand, if $\mu(S^c(x))=1$ for all x, then the net profit always decreases when the covered demand is reduced, so the model cannot discarded.

Definition 4. According to the previous notation, we call validation degree of the fuzzy *p*-median model with respect to the given estimate of the profits to the real number

$$\alpha=\min\mu_x(S^c(x))\in[0,1]$$

We can consider x as the fuzzy truth value of the sentence "there are no virtual losses in the model". The following example illustrates the concepts we have introduce in this section.

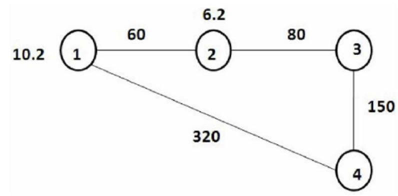
For example, if we consider a network with n vertices, then a vertex with membership degree α is deleted. Now, we solve the problem with fuzzy demands with $n-1$ rest vertices. In this case, desired vertex for decreasing demand, reduction of demand and goal function with membership degree λ will found. In the end of solving *p*-median problem with fuzzy demand total membership degree, that is, membership degree of solution to problem with fuzzy vertices and demand define with following:

$$\gamma=\frac{\lambda+2(1-\alpha)}{2}$$

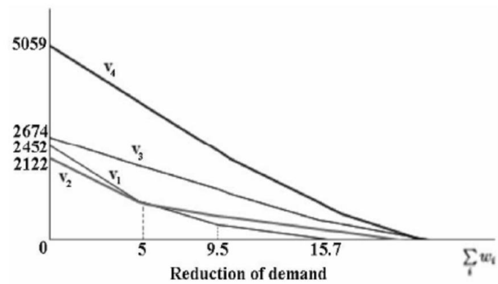
Remark 1. In the end, we notice if we do not consider this ordering, then the problem would not solve or computational complexity increase. Moreover, in median problem with fuzzy vertex, for simplicity, suppose profit function is linear.

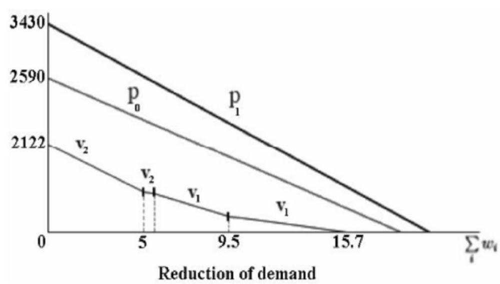
Example 1. As an illustration of our method, we consider a 1-median problem in the four-vertex network given in Figure 1. Suppose net profit function for decrease demand is as follows:

$$p_0(x)=\text{-}229\text{x}+2590,\ \ p_1(x)=\text{-}230.2\text{x}+3430.$$



In according to stated problem, deletion of unnecessary for decreasing of costs, decreasing of demand in a vertex for 1 unit about 100 Euro decreasing in cost and in the end, finding a facility for service of demands of vertices for favorite solution should down.





As seen in the figures, minimum membership degree accrues in $x = 5$, that is, in $v4$ vertex with membership degree α = 0.2. So, the truth value of the existence of losses is 0.8. Therefore, $v4$ is deleted. Now, consider the problem with rest vertices v_1, v_2, v_3 for fuzzy demand problem. We consider

$$p_f = 1.04, p_g = 194.4$$

The solution of problem is given in the table below. Also, general degree of membership is $\lambda = .64$.

References

D. Kutangila-Mayoya and J. L. Verdegay, (2005) p-median problems in a fuzzy environment*, Mathware and Soft Computing 12,* 97-106.

C. Revalle and R. W. Swain, (1970) Central facilities location, *Geographical Analysis 2*, 30-42.

S. L. Hakimi, (1964) Optimum locations of switching centers and the absolute centers and medians of a graph, Oper. Res. 12, 450-469.

M. J. Canos, C. Ivorra and V. Liern, (1999) An exact algorithm for the fuzzy *p*-median problem, European J. Oper. Res. 116, 80-86.

M. J. Canos, C. Ivorra and V. Liern, (2001) The fuzzy *p*-median problem: a global analysis of the solutions, European J. Oper. Res. 130, 430-436.

H. J. Zimmermann, (1997) Fuzzy mathematical programming, Advances in Sensitivity Analysis and Parametric Programming, T. Gol and H. J. Greenbery, eds., Kluwer Academic Publishers, Boston